ON STABILITY OF STATIONARY MOTIONS OF HOLONOMIC AND NONHOLONOMIC SYSTEMS

(OB USTOICHIVOSTI STATSIONARNYKH DVIZHENII GOLONOMNYKH I NEGOLONOMNYKH SISTEM)

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It is shown that dynamic systems with a manifold of steady motions(*) possess a series of features: the corresponding characteristic equation has zero roots; for continuously acting and arbitrarily small disturbances, the motion takes place along this manifold; bifurcations are possible which are unusual for isolated states of equilibrium.(**).

A considerable amount of literature is devoted to the study of steady motions of dynamic systems with cyclic coordinates. The basic results in this area are contained in the works by Routh [2], Klein and Sommerfeld [3], Whittaker [4], Synge [5] and others. According to Whittaker, a steady motion is called such a motion of a system with cyclic coordinates for which the noncyclic coordinates as well as the velocities, corresponding to cyclic coordinates, retain constant values. Routh, Whittaker and others have assumed that in studying the stability of steady motions, one can fully apply the methods which are used in studying the stability of isolated states of equilibrium. By virtue of this, features associated with the presence of a manifold of steady motions remained uninvestigated; the fact itself regarding the nonisolated character of steady states was pointed out by these authors.

In the present paper it is shown that steady motions form a manifold of certain dimensionality which leads to the occurrence of a series of peculiarities. These peculiarities express themselves by the presence of zero roots in the characteristic equation, in the possibility of bifurcation of a new type, which cannot occur for an isolated state of equilibrium, as well as in a special type of behavior of the system for continuously acting small disturbances. Some results of theoretical analysis are illustrated with the aid of an example.

1. Manifold of steady motions. Let $q_1, q_2, \ldots, q_n$ be the generalized coordinates of a holonomic system with Lagrangian function

$$L = L(q_1, \ldots, q_m; \dot{q}_1, \ldots, \dot{q}_m, \dot{q}_{m+1}, \ldots, q_n) \qquad (m < n) \tag{1.1}$$

Here n is a number of generalized coordinates, among which the last $(n-m)$

*) Holonomic system with m cyclic coordinates possesses in the general case a manifold of steady motions of m dimensions.

**) One of such bifurcations was indicated by Ishlinskii in paper [1].

ones are cyclic. Let us consider a system with incomplete dissipation of mechanical energy, for which the Rayleigh

$$F = \frac{1}{2} \sum_{i,j=1}^{m} h_{ij}(q_1, \ldots, q_m) q_i^{\cdot} q_j^{\cdot}$$

does not contain velocities which correspond to cyclic coordinates.

The motion of such a system is described by means of equations

$$\frac{d}{dt}\frac{\partial L}{\partial q_j^{\cdot}} + \sum_{i=1}^{m} h_{ij} q_i^{\cdot} = \frac{\partial L}{\partial q_j}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \omega_k} = 0 \qquad \begin{pmatrix} j = 1, 2, \ldots, m \\ k = 1, \ldots, n-m \end{pmatrix} \quad (1.2)$$

Here

$$\omega_k = q_{m+k}^{\cdot}, \qquad L = L(q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}, \omega_1, \ldots, \omega_{n-m}) \quad (1.3)$$

Equations (1.2) describe the motion of the representative point in $(n+m)$-dimensional space Φ, along the axes of coordinates of which the quantities are measured

$$q_1, q_2, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}, \omega_1, \ldots, \omega_{n-m}$$

By definition, for steady motion

$$q_j = q_j^{\circ} = \text{const}, \qquad q_j^{\cdot} = 0, \qquad \omega_k = \omega_k^{\circ} = \text{const}$$

Consequently, a steady motion is represented in space Φ by a state of equilibrium. Thus, the problem of stability of steady motions is reduced to the problem of stability of states of equilibrium in space Φ. It is apparent that equations

$$\partial L / \partial q_j = 0 \qquad (j = 1, 2, \ldots, m) \quad (1.4)$$

represent a system of m equations with respect to n unknowns $q_1^{\circ}, \ldots, q_m^{\circ}$, $\omega_1^{\circ}, \ldots, \omega_{n-m}^{\circ}$. In as much as $m < n$, it follows directly that in the general case we have a manifold of steady motions of dimension $n-m$.

Let us consider now steady motions of a nonholonomic system. Let the motion of the system just considered be limited by nonholonomic constraints, which are represented by l ($l < n$) equations of the form

$$\sum_{r=1}^{n} a_{sr}(q_1, \ldots, q_m) q_r^{\cdot} = 0 \qquad (s = 1, 2, \ldots l) \quad (1.5)$$

Constructing the equations of motion of the system, we have now instead of (1.2)

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial q_j^{\cdot}} + \sum_{i=1}^{m} h_{ij} q_i^{\cdot} - \frac{\partial L}{\partial q_j} = \sum_{s=1}^{l} \lambda_s a_{sj} \qquad (j = 1, \ldots, m) \\ &\frac{d}{dt}\frac{\partial L}{\partial \omega_k} = \sum_{s=1}^{l} \lambda_s a_{s,m+k} \qquad (k = 1, \ldots, n-m) \\ &\sum_{j=1}^{m} a_{sj} q_j^{\cdot} + \sum_{k=1}^{n-m} a_{s,m+k} \omega_k = 0 \qquad (s = 1, \ldots, l) \end{aligned} \quad (1.6)$$

Here λ_s — are undertermined factors, L is given by Equations (1.3). Equations (1.6) form a system of $(n + l)$ equations for the determination of $n + l$ quantities $q_1, \ldots, q_n$, $\omega_1, \ldots, \omega_{n-m}$, $\lambda_1, \ldots, \lambda_l$ as functions of time.

For steady motion $q_j = q_j^\circ$, $q_j^\cdot = 0$, $\omega_k = \omega_k^\circ$, $\lambda_s = \lambda_s^\circ$. Substituting these values into (1.6), we obtain the equations of steady motions of the nonholonomic system

$$\frac{\partial L}{\partial q_j} + \sum \lambda_s^\circ a_{sj} = 0, \qquad \sum \lambda_s^\circ a_{s,m+k} = 0, \qquad \sum a_{s,m+k}\omega_k^\circ = 0 \qquad (1.7)$$

which form a system of $n + l$ equations for the determination of $n + l$ quantities $q_1^\circ, \ldots q_n^\circ$, $\omega_1^\circ, \ldots, \omega_{n-m}^\circ$, $\lambda_1^\circ, \ldots, \lambda_{n-m}^\circ$. However, not all of these equations are independent.

In fact, if the second equations (1.7) are multiplied, respectively, by ω_k° and are then summed, then by virtue of the third equations (1.7) we obtain identically zero with respect to λ_s°. Consequently, at least one of the equations of the system (1.7) is not independent, i.e. the number of equations for the determination of $q_1^\circ, \ldots, q_n^\circ$, $\omega_1^\circ, \ldots, \omega_{n-m}^\circ$, $\lambda_1^\circ, \ldots, \lambda_l^\circ$, is, at least, one less than the number $(n + l)$ of these quantities.

Thus, both in the case of a holonomic as well as a nonholonomic system, steady motions form a manifold O_q of some dimensionality $q > 0$, whereby in the case of the holonomic system $q \geq n - m$.

Let us write down the equations of motion of the system considered in the normal form

$$\frac{dx_\alpha}{dt} = f_\alpha(x_1, x_2, \ldots, x_{n+m}) \qquad (\alpha = 1, \ldots, n + m) \qquad (1.8)$$

Here x_α indicate $q_1, \ldots, q_n$, $q_1^\cdot, \ldots, q_n^\cdot$, $\omega_1, \ldots, \omega_{n-m}$ A certain q-dimensional surface O_q, determined by equations

$$x_\alpha = x_\alpha^\circ(u_1, u_2, \ldots, u_q) \qquad (1.9)$$

corresponds to the manifold of steady motions in space Φ, where $u_1, \ldots, u_q$ are the running parameters of the surface.*) Along with the variables $u_1, \ldots, u_q$ we introduce new variables $v_1, \ldots, v_{n+m-q}$ by means of the relations

$$x_\alpha = x_\alpha^\circ(u_1, \ldots, u_q) + \sum_{\beta=1}^{n+m-q} \gamma_{\alpha\beta}(u_1, \ldots, u_q) v_\beta \qquad (1.10)$$

Here $\gamma_{\alpha\beta}$ are some functions of the variables $u_1, \ldots, u_q$. In the new variables Equations (1.8) are written down in the form

$$\frac{du_k}{dt} = g_k(u, v), \qquad \frac{dv_r}{dt} = h_r(u, v) \qquad (1.11)$$

We linearize equations of motion (1.11) in the vicinity of the surface O_q.

*) We note that the manifold of steady motions may consist of several components, which may be of varied dimensions. In this case O_q should be taken as one of the components of the manifold of steady motions.

Expanding the right-hand sides of Equations (1.11) into a series in terms of small quantities $v_1, v_2, \ldots, v_{n+m-q}$, we obtain

$$\frac{du_k}{dt} = \sum_{r=1}^{n+m-q} a_{kr}(u_1, \ldots, u_q)\, v_r + O(\|v\|^2) \qquad (k = 1, \ldots, q)$$
$$\frac{dv_r}{dt} = \sum_{s=1}^{n+m-q} b_{rs}(u_1, \ldots, u_q)\, v_s + O(\|v\|^2) \qquad (r = 1, \ldots, n+m-q) \tag{1.12}$$
$$\|v\|^2 = v_1^2 + v_2^2 + \ldots + v_{n+m-q}^2$$

Here $O(\|v\|^2)$ indicates small terms not lower than of second order with respect to $\|v\|$. From (1.12) it follows that for any arbitrary point of the surface O_q the characteristics equation has the form

$$p^q\,|b_{rs} - p\,\delta_{rs}| = 0 \tag{1.13}$$

where δ_{rs} is Kronecker's symbol, from where we immediately obtain q zero roots. Thus, the number of zero roots of the characteristic equation is not smaller than the dimensionality of the manifold of steady motions(*).

2. On the disturbance of steady motion by small continuously acting forces. For the purpose of studying the stability of steady motions with respect to small disturbances of initial conditions, in accordance with what was said above, the theorem on asymptotic stability of the manifold of equilibrium states can be applied, which was formulated in [6].

According to this theorem, asymptotic stability of the manifold of steady motions is completely determined by the roots of Equation (1.13) without taking into account its q zero roots.

In order to stress the peculiarity, which is exhibited in disturbing the steady motion by means of small continuously acting forces, we recall first the known results of the investigation of the behavior of a system which possesses an isolated state of equilibrium [8].

Let the motion of the system be described by Equations

$$\frac{dx_\alpha}{dt} = f_\alpha(x_1, x_2, \ldots, x_n) + \delta_\alpha(t) \qquad (\alpha = 1, \ldots, n) \tag{2.1}$$

Here $\delta_\alpha(t)$ are continuously acting disturbances, which are such that $|\delta_\alpha(t)| < \delta$, where δ is a small positive quantity. Let us assume that at the point $x_1 = x_2 = \ldots x_n = 0$ we have an isolated state of equilibrium.

In describing the behavior of the system in the vicinity of the state of equilibrium we may single out the following cases.

1. State of equilibrium is asymptotically stable in linear approximation. In this case the real parts of all roots of the characteristic equation are

*) The case when the number of zero roots of the characteristic equation is larger than the dimensionality of the manifold of steady motions should be considered as a special one.

negative and for arbitrary, sufficiently small disturbances the quantities $|x(t)|<\varepsilon$, whereby $\varepsilon\to 0$, if $\delta\to 0$.

2. Critical case, but stability is asymptotic, i.e. the real parts of some roots of the characteristic equation are equal to zero, however, there exists a positive definite form V such that, by virtue of the differential equations (2.1) where one has set $\delta_\alpha(t)=0$ the derivative with respect to time dV/dt is negative. In this case the behavior of the system in the vicinity of the isolated state of equilibrium is the same as in the previous case.

3. Critical case, but the stability is not asymptotic. In this case for arbitrarily small continuously acting disturbances the representative point in the space $(x_1,\ldots,x_n)$ may recede from the origin of coordinates by a finite distance.

Let us consider now the case of the disturbance of stationary motions. Let us note first that the presence of a q-dimensional manifold of steady motions, and is not at all a sign that we have to deal here with a critical case of the theory of stability, as this would be the case for an isolated state of equilibrium.

A study of the stability of steady motions is meaningful only with respect to small deviations from surface O_q of steady motions. Thereby it is natural to investigate the second group of Equations (1.12), independently from the first group, temporarily treating the variables $u_1,\ldots,u_q$ as parameters.

Let us assume that in a certain region G of values $u_1,\ldots,u_q$ the state of equilibrium $v_1=v_2=\ldots=v_{n+m-q}=0$ of the system of equations

$$\frac{dv_r}{dt}=\sum_{s=1}^{n+m-q} b_{rs}(u_1,\ldots,u_q)\,v_s \qquad (r=1,\ldots,n+m-q) \tag{2.2}$$

is asymptotically stable, so that

$$\|v\|<M\|v^\circ\|\exp(-\sigma t)$$

Here v_r° are the initial values of the variables

$$v_r\ (r=1,\ \ldots,\ n+m-q),\qquad \sigma>0,\qquad M<+\infty$$

Let the initial values $u_1^\circ,\ldots,u_q^\circ,v_1^\circ\ldots,v^\circ_{n+m-q}$ be such that the values $u_1^\circ,\ldots,u_q^\circ$ lie within the region G of asymptotic stability of Equations (2.2), while the quantities $v_1^\circ,\ldots,v_{n+m-q}^\circ$ are sufficiently small; then, in accordance with the theorem [6] on the asymptotic stability of the manifold of states of equilibrium, by virtue of equations of motion (1.11), the following limit relations are valid

$$\lim_{t\to+\infty} v_r(t)=0,\qquad \lim_{t\to+\infty} u_s(t)=u_s^*$$

Thereby for variables $v_r(t)$ we have the estimate

$$\|v(t)\|<M'\|v^\circ\|\exp(-\sigma' t)\qquad (\sigma>\sigma'>0,\ M'<+\infty)$$

In the presence of an asymptotically stable manifold of steady motions there is valid the following theorem on the behavior of the system in the vicinity of this manifold for small continuously acting disturbances.

Theorem. In the region G of asymptotic stability of steady motions for arbitrary $\varepsilon>0$ we can find such $\delta>0$, that for arbitrary continuously acting disturbances, smaller than δ, the phase point as long as its

u-components are in G, does not leave the ϵ-vicinity of the manifold of steady motions, and we can always find such arbitrarily small continuously acting disturbances, for which the phase point will be displaced on the surface of steady motions along an arbitrary prescribed curve in region G.

From this theorem it follows, in particular, that the connecting branch of the manifold of steady motions is stable with respect to sufficiently small continuously acting disturbances, when all its points belong to the region of asymptotic stability, and unstable, if this connecting branch contains a region of instability.

In the presence of continuously acting disturbances, the equations of motion in the vicinity of the manifold O_q of steady motions may be written in the form

$$\frac{du_k}{dt} = \Sigma a_{kr}(u_1, \ldots, u_q)\, v_r + O(\|v\|^2) + \delta_k(t) \qquad (2.3)$$
$$\frac{dv_s}{dt} = \Sigma b_{sr}(u_1, \ldots, u_q)\, v_r + O(\|v\|^2) + \delta_s(t)$$

To prove the first assertion of the theorem, we use the function of Liapunov [7] for the second group of Equations (2.3). In the region G there exists a positive quadratic form

$$V(\alpha\|v\|^2 \leqslant V \leqslant \beta\|v\|^2, \qquad \alpha > 0, \qquad \beta < \infty)$$

with coefficients, which depend on $u_1, \ldots, u_q$, such that by virtue of (2.2),

$$\frac{dV}{dt} = -\|v\|^2, \qquad \sum_{k=1}^{q}\left|\frac{\partial V}{\partial u_k}\right| < a\|v\|^2 \qquad (a < +\infty)$$

On the surface of steady motions $V=0$ and by virtue of equations (2.3)

$$\frac{dV}{dt} \leqslant -\|v\|^2 + A\|v\|^3 + B\|v\|\delta \qquad (2.4)$$
$$(\delta = \sup\{|\delta_k(t)|,\ |\delta_s(t)|\})$$

From (2.4) it follows that

$$dV/dt < 0 \qquad \text{for} \quad 2B\delta < \|v\| < 1/2A \qquad (2.5)$$

This means that for sufficiently small δ the phase point, whose motion is described by Equations (2.3), will not leave the $2B\delta$-vicinity of the surface of steady motions as long as the variables $u_1, \ldots, u_q$ belong to the region G. Thus the first assertion of the theorem is proved.

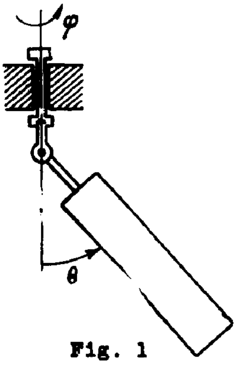

Fig. 1

To prove the second assertion, it is sufficient to establish the fact that one can select arbitrarily small disturbances for which the quantity u_1 will increase all the time, while the remaining variables u_k $(k=2,3,\ldots,q)$ retain constant values. Let $\delta_s(t)=0$, $u_k^{\cdot}=0$ $(k=2,\ldots,q,\ s=1,\ldots,n+m-1)$ and $u_1^{\cdot}=\delta_1>0$; then from (2.3) for v_s it follows:

$$\|v\| < M\|v^\circ\|e^{-\sigma' t} \qquad (0 < \sigma' < \sigma) \qquad (2.6)$$

and from the first q equations we find the required disturbances

$$\delta_1(t) = \delta_1 - \sum_{r=1}^{n+m-q} a_{1r} v_r - O(\|v\|^2), \qquad (2.7)$$

$$\delta_k(t) = -\sum_{r=1}^{n+m-q} a_{kr} v_r - O(\|v\|^2) \qquad (2.7)\ \text{cont.}$$

By virtue of (2.6), as δ_1 and $\|v^\circ\|$ decrease, these disturbances may be made arbitrarily small.

3. **Example: Rotating plane pendulum.** Let us consider the motion of a heavy axially-symmetric body, suspended from a plane hinge, which is fixed in a vertical bearing (Fig.1). We shall neglect the friction in the bearing, while the friction in the hinge is assumed to be viscous. Let θ and φ be the generalized coordinates of the system, C the axial moment of inertia of the body, A the equatorial moment of inertia with respect to the axis of the suspension of the body, l the distance from the axis of the hinge to the center of mass of the body, m the mass of the body, g the acceleration of gravity. We form the expressions of the Lagrangian function L and the dissipation function F:

$$L = {}^1/_2\,[A\,(\theta'^2 + \varphi'^2 \sin^2\theta) + C\varphi'^2 \cos^2\theta] + \text{mg}\, l \cos\theta, \qquad F = {}^1/_2 h\theta'^2$$

where h is the coefficient of viscous friction in the hinge. Let us introduce the dimensionless quantities

$$\tau = t\left(\frac{mgl}{A}\right)^{1/2}, \qquad \alpha = \frac{A-C}{A}, \qquad \beta = \frac{h}{\sqrt{Amgl}} \, (-1 < \alpha < 1,\ \beta > 0)$$

Let us write the equations of motion of the system

$$\frac{d^2\theta}{d\tau^2} + \beta\frac{d\theta}{d\tau} - \alpha\omega^2 \sin\theta\cos\theta + \sin\theta = 0, \ \frac{d}{d\tau}[\omega(1-\alpha\cos^2\theta)] = 0 \qquad (3.1)$$

The equation of steady motion is of the form

$$(\alpha\omega^2\cos\theta - 1)\sin\theta = 0 \qquad (3.2)$$

It follows from here that the manifold of steady motions is one-dimensional and is composed of three branches: 1) $\theta=0$, 2) $\theta=\pi$, 3) $\alpha\omega^2\cos\theta=1$. Since $0\le\theta\le\pi$, $0\le\omega^2<\infty$, we let $\Omega=\omega^2$ and investigate the first quadrant of the plane (Ω,θ). As the parameter α varies from -1 to $+1$, the third branch being in the region $\frac{1}{2}\pi<\theta\le\pi$, as $\alpha\to -0$, recedes into infinity and then, for $\alpha>0$, reappears from infinity in the region $0\le\theta<\frac{1}{2}\pi$. To investigate the stability of the branches of steady motions we form the characteristic equation of the system

$$p\,[(p^2 + \beta p - \alpha\Omega\cos 2\theta + \cos\theta)(1 - \alpha\cos^2\theta) + \alpha^2\Omega\sin^2 2\theta] = 0$$

The zero root is due to the one-dimensionality of the manifold of steady motions. The stability of this manifold is determined by the roots of Equation

$$(1 - \alpha\cos^2\theta)(p^2 + \beta p - \alpha\Omega\cos 2\theta + \cos\theta) + \alpha^2\Omega\sin^2 2\theta = 0 \qquad (3.3)$$

Substituting $\theta=0$, we obtain the characteristic equation for the branch 1

$$p^2 + \beta p - \alpha\Omega + 1 = 0$$

From here it follows that for $\alpha\Omega<1$, the branch 1 is asymptotically stable. In the case when $\alpha\le0$, the branch 1 is always stable. For $\alpha>0$ the branch 1 is stable only in the region $\Omega<\alpha^{-1}$.

Substituting the value $\theta=\pi$ into (3.3), we obtain the characteristic equation for branch 2. The condition of asymptotic stability of branch 2 is of the form $\alpha\Omega+1<0$. Consequently, in the case $\alpha\ge0$ the branch 2 is always unstable, and in the case $\alpha<0$ the branch 2 is stable only in region $\Omega>|\alpha|^{-1}$.

The characteristic equation for branch 3 is

$$(1-\alpha\cos^2\theta)p^2+\beta\,(1-\alpha\cos^2\theta)p+(\sec\theta+3\alpha\cos\theta)\sin^2\theta=0\qquad(\alpha\,\Omega\cos\theta=1)$$

The condition of asymptotic stability of branch 3 is: $\sec\theta+3\alpha\cos\theta>0$. It follows from here that in the case $\alpha>0$ the branch 3 is always stable, while in the case $\alpha<0$ it is unstable if $|\alpha|\leq 1/3$. For the value of α in region $-1<\alpha<-1/3$ a region of stability $|\alpha|^{-1}<\Omega<\sqrt{3|\alpha|^{-1}}$ appears on the branch 3. Outside this interval of values of Ω, the branch 3 remains unstable. The results obtained are shown on Fig. 2, where little circles indicate stable steady motions, and little crosses unstable motions.

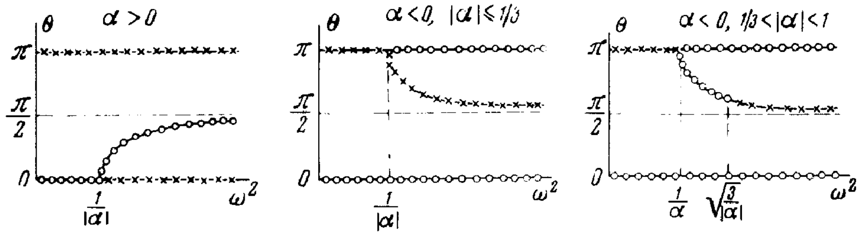

Fig. 2,a,b,c.

The study carried out indicates that the case of bifurcation, which is analogous to that noted by Ishlinskii [1] with the example of a rotating body suspended by a string, is also obtained for the case of a rotating plane pendulum for values of parameter $\alpha<0$ in the region $1/3<|\alpha|<1$, (see fig.2c).

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